



Some inequalities for the s -Godunova–Levin type functions

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Abstract In this paper, first, we obtained two new s -Godunova–Levin type inequalities about “the mean value Theorem for integrals”. Second, some inequalities were proved for mappings q th powers of first derivatives belonging to class $Q_s(I)$ using the Čebyšev’s inequality, Hölder inequality, Power mean inequality, and some other classical inequalities. Finally, some error estimates for the Trapezoidal formula are also given. The results obtained are consistent with literature.

Keywords Hadamard’s inequality · Differentiable mappings · s -Godunova–Levin type functions · Fractional integral · Power mean inequality

Introduction

One of the most famous inequalities for convex functions is Hadamard’s inequality. This double inequality is stated as follows (see for example [1, 2]): Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}. \quad (1)$$

For several recent results concerning the inequality (1), we refer the interested reader to [1–5].

Definition 1 [6] We say that $f : I \rightarrow \mathbb{R}$ is a Godunova–Levin function or that f belongs to class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0, 1)$ we have

$$f(tx + (1-t)y) \leq \frac{1}{t}f(x) + \frac{1}{1-t}f(y). \quad (2)$$

Some further properties of this class of functions can be found in [7–12]. Among others, it has been noted that non-negative monotone and non-negative convex functions belong to this class of functions. The above concept can be extended for functions $f : C \subseteq X \rightarrow [0, \infty)$ where C is a convex subset of the real or complex linear space X and the inequality (2) is satisfied for any vectors $x, y \in C$ and $t \in (0, 1)$. If the function $f : C \subseteq X \rightarrow \mathbb{R}$ is non-negative and convex, then it is of Godunova–Levin type.

Definition 2 Let s be a real number, $s \in (0, 1]$. A function $f : [0, \infty) \rightarrow [0, \infty)$ is said to be s -convex (in the second sense)

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$.

For some properties of this class of functions see [13–17].

This concept can be extended for functions defined on convex subsets of linear spaces in the same way as above replacing the interval I by the corresponding convex subset C of the linear space X :

Definition 3 [18] We say that the function $f : C \subseteq X \rightarrow [0, \infty)$ is of s -Godunova–Levin type, with $s \in [0, 1]$, if

$$f(tx + (1-t)y) \leq \frac{1}{t^s}f(x) + \frac{1}{(1-t)^s}f(y)$$

for all $t \in (0, 1)$ and $x, y \in C$.

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We denote by $\mathcal{Q}_s(C)$ the class of s -Godunova–Levin functions defined on C .

We observe that for $s = 0$, we obtain the class of p -functions while for $s = 1$ we obtain the class of Godunova–Levin. Thus,

$$P(C) = \mathcal{Q}_0(C) \subseteq \mathcal{Q}_{s_1}(C) \subseteq \mathcal{Q}_{s_2}(C) \subseteq \mathcal{Q}_1(C) = \mathcal{Q}(C)$$

for $0 \leq s_1 \leq s_2 \leq 1$.

We recall the well-known Hölder's integral inequality which can be stated as follows, see [19].

Theorem 1 Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on $[a, b]$ and if $|f|^p$ and $|g|^q$ are integrable functions on $[a, b]$, then

$$\int_a^b |f(x)g(x)| dx \leq \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q dx \right)^{\frac{1}{q}}$$

with equality holding if and only if $A|f(x)|^p = B|g(x)|^q$ almost everywhere, where A and B are constants.

Theorem 2 (Power Mean Inequality, see [20]) Let $x = (x_i)$, $p = (p_i)$ be two positive n -tuples and let $r \in \mathbb{R} \cup \{+\infty, -\infty\}$, $i = 1, 2, \dots, n$. Then, taking $p_n = \sum_{k=1}^n p_k$, the r th power mean of x with weights p is defined by

$$M_n^{[r]} = \begin{cases} \left(\frac{1}{p_n} \sum_{k=1}^n p_k x_k^r \right)^{\frac{1}{r}}, & r \neq +\infty, 0, -\infty \\ \left(\prod_{k=1}^n x_k^{p_k} \right)^{\frac{1}{p_n}}, & r = 0 \\ \min(x_1, x_2, \dots, x_n), & r = -\infty \\ \max(x_1, x_2, \dots, x_n), & r = +\infty \end{cases}$$

Note that if $-\infty \leq r < s \leq \infty$, then

$$M_n^{[r]} \leq M_n^{[s]}$$

(see, e.g., [21]).

Theorem 3 [1] Let $f \in \mathcal{Q}(I)$, $a, b \in I$ with $a < b$ and $f \in L_1[a, b]$. Then one has the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{4}{b-a} \int_a^b f(x) dx. \quad (3)$$

Theorem 4 [1] Let $f \in P(I)$, $a, b \in I$ with $a < b$ and $f \in L_1[a, b]$. Then one has the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a} \int_a^b f(x) dx \leq 2(f(a) + f(b)). \quad (4)$$

Both inequalities are best possible.

We need the following inequalities:

Theorem 5 (see [21]) Let $f, g : [a, b] \rightarrow \mathbb{R}$ be integrable functions, both increasing or both decreasing. Furthermore, let $p : [a, b] \rightarrow \mathbb{R}_+$ be an integrable function. Then

$$\int_a^b p(x)f(x) dx \int_a^b p(x)g(x) dx \leq \int_a^b p(x) dx \int_a^b p(x)f(x)g(x) dx. \quad (5)$$

If one of the functions f or g is nonincreasing and the other nondecreasing then the inequality in (5) is reversed. Inequality (5) is known in the literature as Čebyšev's inequality and so are the following special cases of (5):

$$\frac{1}{b-a} \int_a^b f(x) dx \int_a^b g(x) dx \leq \int_a^b f(x)g(x) dx$$

and

$$\int_0^1 f(x) dx \int_0^1 g(x) dx \leq \int_0^1 f(x)g(x) dx.$$

Now, we are giving some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper, see [22].

Definition 4 Let $f \in L_1[a, b]$. The Riemann–Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively, where $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$. Here is $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

For some recent results connected with fractional integral inequalities see [22–29].

In [30, Özdemir et al. proved the following result for fractional integrals.

Lemma 1 Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I with $a < r$, $a, r \in I$. If $f' \in L[a, r]$, then the following equality for fractional integrals holds:

$$\begin{aligned} & \frac{f(a) + f(r)}{2} - \frac{\Gamma(\alpha+1)}{2(r-a)^\alpha} [J_{r-}^\alpha f(a) + J_{a+}^\alpha f(r)] \\ &= \frac{r-a}{2} \int_0^1 [(1-t)^\alpha - t^\alpha] f'(r + (a-r)t) dt. \end{aligned}$$



The main purpose of this study is to obtain the inequalities for class of s -Godunova–Levin type functions.

Main results

In [31], S. S. Dragomir has written the following inequalities (6) and (8) without proof.

Theorem 6 Let $f \in Q_s(C)$, with $a < b$ and $f \in L_1[a, b]$, $C = [a, b]$, $s \in [0, 1]$. Then one has the inequalities

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{s+1}}{b-a} \int_a^b f(x) dx \quad (6)$$

$$\frac{\Gamma((1+s)^2)}{\Gamma(2+2s)(b-a)} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{s+1} \quad (7)$$

and

$$\frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{1-s}, \quad s \in [0, 1] \quad (8)$$

Proof $f \in Q_s(C)$, we have for all $x, y \in C = [a, b]$ with $t = \frac{1}{2}$;

$$2^s(f(x) + f(y)) \geq f\left(\frac{x+y}{2}\right)$$

Now, if we choose $x = ta + (1-t)b$, $y = (1-t)a + tb$, we have

$$2^s(f(ta + (1-t)b) + f((1-t)a + tb)) \geq f\left(\frac{a+b}{2}\right).$$

By integrating, we have that

$$2^s \left[\int_0^1 f(ta + (1-t)b) dt + \int_0^1 f((1-t)a + tb) dt \right] \geq f\left(\frac{a+b}{2}\right) \quad (9)$$

On the other hand,

$$\begin{aligned} \int_0^1 f(ta + (1-t)b) dt &= \int_0^1 f((1-t)a + tb) dt \\ &= \frac{1}{b-a} \int_a^b f(x) dx \end{aligned}$$

we get the inequality (6) from (9).

For the proof of (7), if $f \in Q_s(C)$ for all $a, b \in C$ and $t \in (0, 1)$, it yields

$$t^s(1-t)^s f(ta + (1-t)b) \leq (1-t)^s f(a) + t^s f(b)$$

and

$$t^s(1-t)^s f((1-t)a + tb) \leq t^s f(a) + (1-t)^s f(b).$$

By adding these inequalities and integrating over $[0, 1]$, we find that

$$\begin{aligned} &\int_0^1 t^s(1-t)^s [f(ta + (1-t)b) + f((1-t)a + tb)] dt \\ &\leq \frac{2}{s+1} [f(a) + f(b)]. \end{aligned}$$

Now, by a simple computation, we have

$$\int_0^1 t^s(1-t)^s f(ta + (1-t)b) dt \text{ and } \int_0^1 t^s(1-t)^s f((1-t)a + tb) dt.$$

Let be $g(t) = t^s(1-t)^s$. We take symmetric of the functions $g(t)$ and f , respectively, $\frac{1}{2}$ and $\frac{a+b}{2}$. Also, let the functions f and g both be either increasing or decreasing. By applying Čebyšev's inequality, we have

$$\begin{aligned} \int_0^1 t^s(1-t)^s f(ta + (1-t)b) dt &= \int_0^1 t^s(1-t)^s f((1-t)a + tb) dt \\ &\geq \int_0^1 t^s(1-t)^s dt \int_0^1 f((1-t)a + tb) dt \\ &= \frac{\Gamma((1+s)^2)}{\Gamma(2+2s)(b-a)} \int_a^b f(x) dx. \end{aligned}$$

To obtain the inequality (8), as $f \in Q_s(C)$, we have

$$f(ta + (1-t)b) \leq t^{-s} f(a) + (1-t)^{-s} f(b)$$

integrating this inequality on $[0, 1]$, we get

$$\begin{aligned} \int_0^1 f(ta + (1-t)b) dt &\leq f(a) \int_0^1 t^{-s} dt + f(b) \int_0^1 (1-t)^{-s} dt \\ &= \frac{f(a)+f(b)}{1-s}, \quad (s \in [0, 1)) \end{aligned}$$

As the change of variable $x = ta + (1-t)b$ gives us

$$\int_0^1 f(ta + (1-t)b) dt = \frac{1}{b-a} \int_a^b f(x) dx$$

which completes the proof of the inequality (8). \square

Theorem 7 Combining the inequalities (2.1) and (2.3) under the conditions of Theorem 6, we get

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{s+1}}{b-a} \int_a^b f(x) dx \leq 2^{s+1} \frac{f(a)+f(b)}{1-s}.$$

Remark 1 If we choose $s = 1$ in (6) and (7), we obtain the inequality (3) and right hand side of (4), respectively.

Theorem 8 Let $f : I \subseteq \mathbb{R} \rightarrow [0, \infty)$ be a differentiable mapping on I , $a, r \in I$ and $a < r$. If $|f'| \in Q_\alpha(I)$ with $\alpha \in [0, 1]$, $t \in (0, 1)$, then the following inequality for fractional integrals holds:

$$\begin{aligned} &\left| \frac{f(a)+f(r)}{2} - \frac{\Gamma(\alpha+1)}{2(r-a)^\alpha} [J_{r-}^\alpha f(a) + J_{a+}^\alpha f(r)] \right| \\ &\leq \frac{(r-a)}{2} \left\{ \pi \alpha \operatorname{cosec}(\pi \alpha) [|f'(r)| + |f'(a)|] + 2\beta_{\frac{1}{2}}(1-\alpha, 1+\alpha) \right\} \quad (10) \end{aligned}$$



where $\beta_x(y, z) = \int_0^x t^{y-1}(1-t)^{z-1} dt$, $0 \leq x \leq 1$ is incomplete Beta function.

Proof Using the Lemma 1 and $|f'| \in Q_\alpha(I)$, it follows that

$$\left| \frac{f(a) + f(r)}{2} - \frac{\Gamma(\alpha + 1)}{2(r-a)^\alpha} [J_r^\alpha f(a) + J_{a^+}^\alpha f(r)] \right| \leq \frac{(r-a)}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| |f'(r + (a-r)t)| dt.$$

Since

$$|f'(r + (a-r)t)| = |f'(ta + (1-t)r)| \leq \frac{1}{t^\alpha} |f'(a)| + \frac{1}{(1-t)^\alpha} |f'(r)|$$

$$\begin{aligned} & \left| \frac{f(a) + f(r)}{2} - \frac{\Gamma(\alpha + 1)}{2(r-a)^\alpha} [J_r^\alpha f(a) + J_{a^+}^\alpha f(r)] \right| \\ & \leq \frac{(r-a)}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| \left[\frac{1}{t^\alpha} |f'(a)| + \frac{1}{(1-t)^\alpha} |f'(r)| \right] dt \\ & = \frac{(r-a)}{2} \left\{ \int_0^{\frac{1}{2}} [(1-t)^\alpha - t^\alpha] \left[\frac{1}{t^\alpha} |f'(a)| + \frac{1}{(1-t)^\alpha} |f'(r)| \right] dt \right. \\ & \quad \left. + \int_{\frac{1}{2}}^1 [t^\alpha - (1-t)^\alpha] \left[\frac{1}{t^\alpha} |f'(a)| + \frac{1}{(1-t)^\alpha} |f'(r)| \right] dt \right\} \end{aligned}$$

On the other hand, by a simple computation, we have

$$\begin{aligned} \int_0^{\frac{1}{2}} \left[\frac{(1-t)^\alpha - t^\alpha}{t^\alpha} \right] dt &= \beta_{\frac{1}{2}}(1-\alpha, 1+\alpha) - \frac{1}{2} \\ \int_0^{\frac{1}{2}} \left[\frac{(1-t)^\alpha - t^\alpha}{(1-t)^\alpha} \right] dt &= \frac{1}{2} - \beta_{\frac{1}{2}}(\alpha+1, 1-\alpha) \\ \int_{\frac{1}{2}}^1 \left[\frac{t^\alpha - (1-t)^\alpha}{t^\alpha} \right] dt &= \frac{1}{2} - \pi \alpha \operatorname{cosec}(\pi \alpha) + \beta_{\frac{1}{2}}(\alpha+1, 1-\alpha) \end{aligned}$$

and

$$\int_{\frac{1}{2}}^1 \left[\frac{t^\alpha - (1-t)^\alpha}{(1-t)^\alpha} \right] dt = \pi \alpha \operatorname{cosec}(\pi \alpha) - \beta_{\frac{1}{2}}(1+\alpha, 1-\alpha) - \frac{1}{2}.$$

Since, finally

$$|f'(r)| - |f'(a)| \leq |f'(r)| + |f'(a)|.$$

Hence, we obtain the inequality (10). \square

Theorem 9 Let $f : I \subset \mathbb{R} \rightarrow [0, \infty)$ be a differentiable mapping of I , $a, r \in I$ and $a < r$. If $|f'|^q \in Q_s(I)$ with $\alpha \in [0, 1]$, $t \in (0, 1)$, then the following inequality for fractional integrals holds:

$$\begin{aligned} & \left| \frac{f(a) + f(r)}{2} - \frac{\Gamma(\alpha + 1)}{2(r-a)^\alpha} [J_r^\alpha f(a) + J_{a^+}^\alpha f(r)] \right| \\ & \leq \frac{(r-a)}{2(\alpha p + 1)^{\frac{1}{p}}(1-s)^{\frac{1}{q}}} (|f'(a)|^q + |f'(r)|^q)^{\frac{1}{q}} \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $s \in [0, 1]$, $\Gamma(\cdot)$ is Gamma function.

Proof From Lemma 1 and using Hölder inequality with properties of modulus, we have

$$\begin{aligned} & \left| \frac{f(a) + f(r)}{2} - \frac{\Gamma(\alpha + 1)}{2(r-a)^\alpha} [J_r^\alpha f(a) + J_{a^+}^\alpha f(r)] \right| \\ & \leq \frac{(r-a)}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| |f'(r + (a-r)t)| dt \\ & \leq \frac{(r-a)}{2} \left(\int_0^1 |(1-t)^\alpha - t^\alpha|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(r + (a-r)t)|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

We know that for $\alpha \in (0, 1]$ and $\forall t_1, t_2 \in (0, 1)$,

$$|t_1^\alpha - t_2^\alpha| \leq |t_1 - t_2|^\alpha,$$

therefore,

$$\begin{aligned} \int_0^1 |(1-t)^\alpha - t^\alpha|^p dt &\leq \int_0^1 |1-2t|^{2p} dt \\ &= \int_0^{\frac{1}{2}} (1-2t)^{2p} dt + \int_{\frac{1}{2}}^1 (2t-1)^{2p} dt \\ &= \frac{1}{\alpha p + 1}. \end{aligned}$$

Since $|f'| \in Q_s(I)$, we obtain

$$|f'(r + (a-r)t)| = |f'(ta + (1-t)r)| \leq \frac{1}{t^s} |f'(a)| + \frac{1}{(1-t)^s} |f'(r)|.$$

Hence, we get

$$\begin{aligned} & \left| \frac{f(a) + f(r)}{2} - \frac{\Gamma(\alpha + 1)}{2(r-a)^\alpha} [J_r^\alpha f(a) + J_{a^+}^\alpha f(r)] \right| \\ & \leq \frac{(r-a)}{2(\alpha p + 1)^{\frac{1}{p}}(1-s)^{\frac{1}{q}}} (|f'(a)|^q + |f'(r)|^q)^{\frac{1}{q}} \end{aligned}$$

which completes the proof. \square

Corollary 1 In Theorem 9, if we choose $\alpha = 1$ and $s = 0$, then we have

$$\begin{aligned} & \left| \frac{f(a) + f(r)}{2} - \frac{1}{(r-a)} \int_a^r f(t) dt \right| \\ & \leq \frac{(r-a)}{2(p+1)^{\frac{1}{p}}(1-s)^{\frac{1}{q}}} (|f'(a)|^q + |f'(r)|^q)^{\frac{1}{q}} \\ & \leq \frac{(r-a)}{2} (|f'(a)| + |f'(r)|). \end{aligned} \quad (11)$$

Proof Let $a_1 = |f'(a)|^q$, $b_1 = |f'(r)|^q$, $0 < \frac{1}{q} < 1$ for $q > 1$. Using the fact

$$\sum_{i=1}^n (a_i + b_i)^r \leq \sum_{i=1}^n a_i^r + \sum_{i=1}^n b_i^r$$



for $a_1, a_2, \dots, a_n \geq 0$, $b_1, b_2, \dots, b_n \geq 0$, we obtain the inequality (11) and since $\frac{1}{2} \leq (\frac{1}{p+1})^{\frac{1}{p}} \leq 1$, $p \in (0, 1)$. \square

Theorem 10 Let $f : I \subset \mathbb{R} \rightarrow [0, \infty)$ be a differentiable mapping on I , $a, r \in I$, $a < r$ and $q \geq 1$. If $|f'|^q \in Q_\alpha(I)$ with $\alpha \in [0, 1)$, $t \in (0, 1)$, then the following inequality for fractional integrals hold:

$$\begin{aligned} & \left| \frac{f(a) + f(r)}{2} - \frac{\Gamma(\alpha + 1)}{2(r-a)^\alpha} [J_{r-}^\alpha f(a) + J_{a+}^\alpha f(r)] \right| \\ & \leq \frac{(r-a)}{2^\alpha(\alpha+1)^{1-\frac{1}{q}}} \left(1 - \frac{1}{2^\alpha}\right)^{1-\frac{1}{q}} \left[(2\beta_{\frac{1}{2}}(1-\alpha, 1+\alpha) - \pi\alpha \operatorname{cosec}(\pi\alpha)) \right]^{\frac{1}{q}} \\ & \quad \times (|f'(a)| + |f'(r)|). \end{aligned} \quad (12)$$

Proof From Lemma 1 and using the well-known power mean inequality, we have

$$\begin{aligned} & \left| \frac{f(a) + f(r)}{2} - \frac{\Gamma(\alpha + 1)}{2(r-a)^\alpha} [J_{r-}^\alpha f(a) + J_{a+}^\alpha f(r)] \right| \\ & \leq \frac{(r-a)}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| |f'(r + (a-r)t)| dt \\ & \leq \frac{(r-a)}{2} \left(\int_0^1 |(1-t)^\alpha - t^\alpha| dt \right)^{1-\frac{1}{q}} \\ & \quad \times \left(\int_0^1 |(1-t)^\alpha - t^\alpha| |f'(r + (a-r)t)|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \int_0^1 |(1-t)^\alpha - t^\alpha| dt &= \int_0^{\frac{1}{2}} [(1-t)^\alpha - t^\alpha] dt + \int_{\frac{1}{2}}^1 [t^\alpha - (1-t)^\alpha] dt \\ &= \frac{2}{\alpha+1} \left(1 - \frac{1}{2^\alpha}\right). \end{aligned}$$

Since $|f'|^q \in Q_\alpha(I)$, we have

$$\begin{aligned} |f'(r + (a-r)t)|^q &= |f'(ta + (1-t)r)|^q \leq \frac{1}{t^\alpha} |f'(a)|^q \\ &+ \frac{1}{(1-t)^\alpha} |f'(r)|^q \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 |(1-t)^\alpha - t^\alpha| |f'(r + (a-r)t)|^q dt \\ & \leq \int_0^1 |(1-t)^\alpha - t^\alpha| \left[\frac{1}{t^\alpha} |f'(a)|^q + \frac{1}{(1-t)^\alpha} |f'(r)|^q \right] dt \\ & \leq \int_0^{\frac{1}{2}} [(1-t)^\alpha - t^\alpha] \left[\frac{1}{t^\alpha} |f'(a)|^q + \frac{1}{(1-t)^\alpha} |f'(r)|^q \right] dt \\ & \quad + \int_{\frac{1}{2}}^1 [t^\alpha - (1-t)^\alpha] \left[\frac{1}{t^\alpha} |f'(a)|^q + \frac{1}{(1-t)^\alpha} |f'(r)|^q \right] dt. \end{aligned}$$

and since $\sum_{i=1}^n (a_i + b_i)^r \leq \sum_{i=1}^n a_i^r + \sum_{i=1}^n b_i^r$, we obtain the required inequality (12). \square

Applications to numerical integration

We may not be given a formula for $f(x)$ as a function of x . For instance, $f(x)$ may be an unknown function whose values are at certain points of the interval $[a, b]$. In this case, we investigate the problem of approximating the value of the integral $I = \int_a^b f(x) dx$ using only the values of $f(x)$ at finitely many points of $[a, b]$. Obtaining such an approximation is called numerical integration. That is why, there are three methods for evaluating definite integrals numerically. One of them is Trapezoid Rule.

Let d be a division of the interval $[a, r]$, i.e., $d : a = x_0 < x_1 < \dots < x_{n-1} < x_n = r$, and consider the trapezoidal formula

$$T_n(f, d) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i).$$

So, the following approximation of the integral $\int_a^b f(x) dx$ holds:

$$\int_a^b f(x) dx \approx T_n(f, d) + E_n(f, d)$$

where the approximation error $E_n(f, d)$ of the integral $\int_a^b f(x) dx$ by the trapezoidal formula $T_n(f, d)$ satisfies

$$|E_n(f, d)| \leq \frac{M}{12} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3.$$

We shall propose some new estimates of the remainder term $E_n(f, d)$.

Proposition 1 Let f be a differentiable mapping on I° , $a, r \in I^\circ$ with $a < r$. If $|f'|$ is p -convex on $[a, r]$, then for every division d of $[a, r]$, the following holds:

$$\begin{aligned} |E_n(f, d)| &\leq \frac{1}{2} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 (|f'(x_i)| + |f'(x_{i+1})|) \\ &\leq \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \max\{|f'(a)|, |f'(r)|\}. \end{aligned}$$

Proof Applying Corollary 1 on the subinterval $[x_i, x_{i+1}]$ ($i = 0, 1, 2, \dots, n-1$) of the division d , we get

$$\begin{aligned} & \left| \frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i) - \int_{x_i}^{x_{i+1}} f(x) dx \right| \\ & \leq \frac{(x_{i+1} - x_i)^2 (|f'(x_i)| + |f'(x_{i+1})|)}{2}. \end{aligned}$$

Summing over i from 0 to $n-1$ on taking into account that $|f'|$ is p -convex, we deduce, by the triangle inequality that



$$\begin{aligned}
\left| T_n(f, d) - \int_a^b f(x) dx \right| &\leq \frac{1}{2} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 (|f'(x_i)| + |f'(x_{i+1})|) \\
&\leq \max\{|f'(x_i)|, |f'(x_{i+1})|\} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \\
&\leq \max\{|f'(a)|, |f'(r)|\} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2.
\end{aligned}$$

□

Proposition 2 Let f be a differentiable mapping on $I^\circ \subset I$, $a, r \in I^\circ$ with $a < r$ and let $\frac{1}{p} + \frac{1}{q} = 1$. If $|f'|^q \in Q_s(I^\circ)$ with $\alpha = 1$, $t \in (0, 1)$. Then for every division of $[a, r]$, the following holds:

$$\begin{aligned}
|E_n(f, d)| &\leq \frac{1}{2(p+1)^{\frac{1}{q}}(1-s)^{\frac{1}{q}}} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 [|f'(x_i)|^q + |f'(x_{i+1})|^q]^{\frac{1}{q}} \\
&\leq \frac{\max\{|f'(a)|, |f'(r)|\}}{2(p+1)^{\frac{1}{q}}(1-s)^{\frac{1}{q}}} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2.
\end{aligned}$$

Proof If we apply the Theorem 9 for $\alpha = 1$, the proof is similar to that of Proposition 1. □

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